

Rotating Boson Stars in $(2 + 1)$ Dimensions

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ABSTRACT

We consider rotating boson star solutions in a three-dimensional anti-de Sitter spacetime and investigate the influence of the rotation on their properties. The mass and angular momentum of these configurations are computed by using the counterterm method. No regular solution is found in the limit of vanishing cosmological constant.

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1 Introduction

The problem of finding globally regular soliton solutions with a nonvanishing angular momentum has recently enjoyed a renewed interest, various physical systems having been considered in the literature. A somewhat unexpected result obtained in this context was the absence of rotating regular solutions with a nonvanishing magnetic charge in a spontaneously broken nonabelian gauge theory [1]-[3].

To our knowledge, the only explicit example of rotating solitons in asymptotically flat space are found in a complex scalar field theory.¹ Within general relativity, the properties of the corresponding rotating boson star (BS) solutions are discussed in refs. [7]-[9]. BS are well-known gravitational bound states of complex scalar fields, providing us with the simplest model of relativistic stars. These objects were first studied by Kaup [10] as well as Ruffini and Bonazzalo [11], and, since then, a large number of papers have been published on this subject, including a number of reviews (see *e.g.* [12]-[14]). Ignoring the effects of gravity, the analogous of BS are Q-balls. These are soliton solutions for a complex scalar field with a non-renormalizable self-interaction. Four dimensional, spinning Q-ball solutions have been recently constructed in ref. [15].

As usual, it is of interest to see how the dimensionality D of spacetime affects the properties of these rotating solutions. The first obvious case is $D = 3$, where the problem simplifies dramatically to solving a set of ordinary differential equations. Also, it would be desirable to have available a lower-dimensional toy-model which could exhibit the key features without unnecessary complication. Gravity in three dimensions has attracted much attention in recent years, since Bañados, Teitelboim and Zanelli (BTZ) found a black-hole spacetime [16], which provides an important testing ground for quantum gravity and AdS/CFT correspondence. Many other types of 3D solutions have also been found by coupling matter fields to gravity in different ways. Nongravitating, spinning Q-balls in a (2+1)-dimensional flat background have been constructed in ref. [15], and present very similar properties to their four-dimensional counterparts (see also ref. [17]).

In this letter we address the problem of finding rotating BS solutions in a three dimensional spacetime.² This is rather special case, since, in a remarkable development, exact BS static solutions with a negative cosmological constant, $\Lambda < 0$, were found in ref. [20], in the limit of large self-interaction (see also ref. [21]). Working in the same limit, rotating BS solutions with $\Lambda < 0$ have been found numerically in ref. [22]. However, the large self-interaction limit considered in ref. [22] makes obscure the influence of a number of physical parameters on the solutions' properties, without significantly simplifying the field equations.

¹Note also the existence of slowly rotating Einstein-Yang-Mills solitons [4]; however, they have been found within a nonperturbative approach in an anti-de Sitter (AdS) spacetime only [5], while their existence in asymptotically flat space is unclear [2, 6].

²Rotating stars in a (2+1)-dimensional AdS spacetime are discussed in refs. [18, 19], and present interesting properties. However, the matter sources of these configurations do not have a field theory interpretation.

Static, circularly-symmetric, non self-interacting BS solutions of the three-dimensional gravity with negative cosmological constant were discussed in a more general context in ref. [23]. Similar to the well-known spherically-symmetric four-dimensional case, these circularly symmetric configurations comprise a two-parameter family, labeled by (ϕ_0, n) , where ϕ_0 is the central value of the scalar field and n is the node number $n = 0, 1, \dots$ of the scalar field. However, there are also major differences, *e.g.* the existence in three dimensions of a maximal allowed value for ϕ_0 and the absence of local extrema for particle number and total mass. Also, no regular solutions are found in the asymptotically flat limit. Here we generalize these static solutions by including a rotating term in the general ansatz.

2 The ansatz and general relations

We consider a complex scalar field Φ with a potential $V(|\Phi|^2)$ minimally coupled to AdS gravity. The corresponding action of the system is

$$S = - \int_{\mathcal{M}} d^3x \sqrt{-g} \left(\frac{1}{16\pi G} (R - 2\Lambda) - (g^{ij} \Phi_{,i}^* \Phi_{,j} + V(|\Phi|^2)) \right) + \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^2x \sqrt{-h} K, \quad (1)$$

where the second term is the Hawking–Gibbons surface term [24]. Here, K is the trace of the extrinsic curvature of the boundary ($\partial\mathcal{M}$ at spatial infinity) and h_{ab} is the induced metric on the boundary, while the asterisk denotes complex conjugation. Throughout this letter we set $c = \hbar = 1$; also, the indices $\{i, j, \dots\}$ will indicate the bulk coordinates and $\{a, b, \dots\}$ will indicate the intrinsic coordinates of the boundary metric. The field equations are obtained by varying the action (1) with respect to the field variables g_{ij} and Φ . They are

$$R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = 8\pi G T_{ij}, \quad (2)$$

$$\left(\nabla^2 - \frac{dV}{d|\Phi|^2} \right) \Phi = 0, \quad (3)$$

where the energy momentum tensor is defined by

$$T_{ij} = \Phi_{,i}^* \Phi_{,j} + \Phi_{,j}^* \Phi_{,i} - g_{ij} (g^{km} \Phi_{,k}^* \Phi_{,m} + V(|\Phi|^2)). \quad (4)$$

Since the theory possesses a global $U(1)$ symmetry, there is a conserved Noether current

$$J^k = i g^{kl} (\Phi_{,l}^* \Phi - \Phi_{,l} \Phi^*), \quad (5)$$

and an associated conserved charge, namely, the number of scalar particles

$$N = \int d^2x \sqrt{-g} J^t. \quad (6)$$

Working in (2+1) dimensions, we consider a metric ansatz of the form

$$ds^2 = \frac{dr^2}{F(r)} + r^2 (d\varphi + \Omega(r) dt)^2 - F(r) e^{-2\delta(r)} dt^2, \quad (7)$$

where, following the D -dimensional ansatz used in ref. [23], we take $F(r) = 1 - 2m(r) + r^2/l^2$ (with $\Lambda = -1/l^2$). The coordinate range is $0 \leq r < \infty$, $-\infty < t < \infty$, while the angular variable φ is assumed to vary between 0 and 2π .

The scalar field ansatz considered here is $\Phi = \phi(r)e^{i(k\varphi - \omega t)}$, where $\phi(r)$ is a real function and ω is a real constant. The uniqueness of the scalar field under a complete rotation $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ requires k to be an integer, which we will call the vorticity number from now on. Models with $k = 0$ corresponds to static, circularly-symmetric configurations discussed in ref. [23]. The expression for the particle number is

$$N = 4\pi \int_0^\infty dr \phi^2 r \frac{e^\delta}{F} (\omega + k\Omega). \quad (8)$$

We also mention the following relation between the total angular momentum \bar{J} and the particle number

$$\bar{J} = kN, \quad (9)$$

which is valid for Kerr-like rotating BS in any dimension $D \geq 3$ and $\Lambda \leq 0$. The above relation has originally been found for four-dimensional asymptotically flat BS [8] and holds also for Q-ball configurations [15]. The angular momentum was defined here as

$$\bar{J} = \frac{1}{8\pi G} \int R_i^j \xi_{(\varphi)}^i d^2 \Sigma_j = \int dr d\varphi \sqrt{-g} T_\varphi^t = 4k\pi \int_0^\infty \phi^2 \frac{e^{2\delta}}{F} (\omega + \Omega k), \quad (10)$$

and agrees with the definition obtained later by using the counterterm prescription. Thus, the angular momentum is quantized, which clearly contrasts with the rotating perfect fluid star solutions discussed in ref. [18].

The results presented in this letter correspond to a simple scalar potential $V(\phi) = \mu^2 \phi^2$, (where μ is the scalar field mass), although the field equations have been integrated also for more complicated forms of V .

Since it is convenient to use dimensionless quantities in numerical computation, we perform the rescalings $r \rightarrow r/\mu$, $\phi \rightarrow \phi/\sqrt{16\pi G}$, $\Lambda \rightarrow \Lambda/\mu^2$, $\Omega \rightarrow \Omega\omega$, while the factor ω/μ is absorbed into the definition of the metric function δ . With these conventions, we find the field equations

$$m' = \frac{1}{4} e^{2\delta} r^3 \Omega'^2 + \frac{1}{2} r F \phi'^2 + \frac{1}{2r} k^2 \phi^2 + \frac{1}{2} r V(\phi) + \frac{1}{2F} e^{2\delta} r (1 + k\Omega)^2 \phi^2, \quad (11)$$

$$(e^{-\delta})' = r \left(e^{-\delta} \phi'^2 + e^\delta (1 + k\Omega)^2 \frac{\phi^2}{F^2} \right), \quad (12)$$

$$(r e^{-\delta} F \phi')' = \frac{1}{2} r e^{-\delta} \frac{\partial V}{\partial \phi} + e^{-\delta} k^2 \frac{\phi}{r} - e^\delta \frac{r}{F} (1 + k\Omega)^2 \phi, \quad (13)$$

$$(r^3 e^\delta \Omega')' = 2k e^\delta \frac{r}{F} (1 + k\Omega) \phi^2, \quad (14)$$

where a prime denotes the derivative with respect to r , while the expression for the energy density is

$$\rho = -T_t^t = (1 - \Omega^2 k^2) e^{2\delta} \frac{\phi^2}{F} + V(\phi) + \frac{k^2 \phi^2}{r^2} + \phi'^2 F. \quad (15)$$

For nonsingular solutions, the boundary conditions at the origin should be $k\phi(0) = 0$, so the scalar field must vanish at the origin for rotating configurations, in contrast with the static $k = 0$ case. At spatial infinity, the scalar field must vanish and the metric approaches a BTZ form. If we assume that this metric possesses a symmetry center (located at $r = 0$), and has no conical singularities, we have to impose the condition $\lim_{r \rightarrow 0} m(r) = 0$ [18], while the function $\Omega(r)$ can be nonzero in the same limit.

For small r , a power series solution gives for $k \neq 0$ (the corresponding expansion for static configurations is given in ref. [23])

$$\begin{aligned}
\phi(r) &= cr^{|k|} + O(r^{|k|+1}), \\
\Omega(r) &= \Omega_0 + \frac{c^2|k|(k\Omega_0 + 1)}{2k(|k| + 1)}r^{2|k|} + O(r^{2|k|+1}), \\
\delta(r) &= \delta_0 - \frac{1}{2}c^2|k|r^{2|k|} + O(r^{2|k|+1}), \\
m(r) &= \frac{c^2|k|}{2}r^{2|k|} + O(r^{2|k|+1}),
\end{aligned} \tag{16}$$

where c is a constant (it is enough to consider $c > 0$ only), while δ_0 and Ω_0 are determined by the behavior in the asymptotic region. Ω_0 corresponds to the angular velocity of the star near its center of rotation.

The analysis of the field equations as $r \rightarrow \infty$ gives

$$\begin{aligned}
\phi(r) &\sim \hat{\phi}_0 r^\alpha + O(r^{\alpha-1}), \\
m(r) &\sim M - \frac{1}{2}\frac{J^2}{r^2} + \frac{\hat{\phi}_0^2}{4(\alpha + 1)}\left(1 + \frac{\alpha^2}{l^2}\right)r^{2(\alpha+1)} + O(r^{2\alpha+1}), \\
e^{-\delta(r)} &\sim 1 + \frac{\hat{\phi}_0^2\alpha}{2}r^{2\alpha} + O(r^{2\alpha-1}), \\
\Omega &\sim \frac{J}{r^2} + \frac{kl^2\hat{\phi}_0^2}{2\alpha(\alpha - 1)}r^{2(\alpha-1)} + O(r^{2\alpha-3}),
\end{aligned} \tag{17}$$

where $\alpha = -1 - \sqrt{1 + l^2}$ and $M, \hat{\phi}_0$ and J are constants. We note that, as in other physical situations involving a massive scalar field [25], a nonzero cosmological constant implies a complicated power decay at infinity, rather than an exponential one, as is found in an asymptotically flat space. However, the asymptotic behavior of the metric is truly AdS (*i.e.* $|g_{tt}| \sim r^2/l^2 + O(r^0)$, without linear terms in r). Therefore, as shown in ref. [26], the asymptotic symmetry group is the conformal one, which contains the AdS group as a subgroup.

3 Numerical results and properties of solutions

So far, we have not found an exact solution of the equations of motion even in the absence of rotation, and the resulting system had, instead, to be solved numerically. Numerical arguments

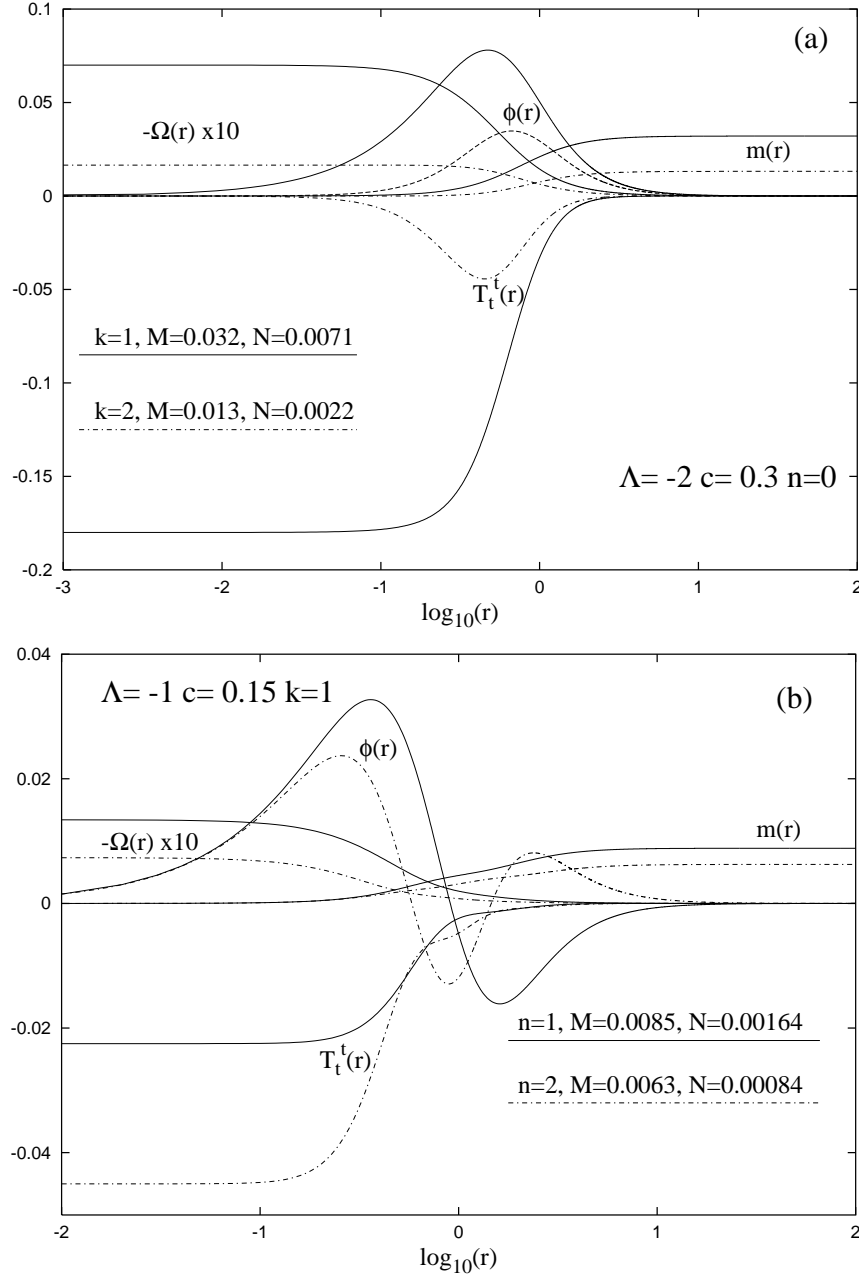


Figure 1. The metric functions $m(r), \Omega(r)$, the scalar field $\phi(r)$ and the energy density are shown as a function of the radial coordinate r for typical solutions with node number $n = 0$ and azimuthal number $k = 1, 2$ (Figure 1a). Higher node solutions with $k = 1$ are presented in Figure 1b. Here and in Figure 2, the particle number N is given in units $1/(4G\mu)$.

for the existence of $k = 0$ static, non self-interacting BS solutions are presented in ref. [23]. Here, we will investigate models with nonzero values of the vorticity number (since the field equations are invariant under $k \rightarrow -k$, $\Omega \rightarrow -\Omega$ it is enough to consider positive values of k only).

Following the usual approach, and by using a standard ordinary differential equation solver, we evaluate the initial conditions (16) at $r = 10^{-5}$ for global tolerance 10^{-12} , adjusting for fixed shooting parameters and integrating towards $r \rightarrow \infty$. We have found that, given (c, k, Λ) , solutions may exist for a discrete set of shooting parameters (δ_0, Ω_0) . Different values of (δ_0, Ω_0) correspond to different numbers, n , of nodes of the scalar field. To simplify the general picture, we consider here nodeless configurations mainly. However, we investigated solutions with up to four nodes and found that they possess a similar behavior to $n = 0$ configurations. Also, the field equations (11)-(14) have been integrated for values of the cosmological constant $0 < |\Lambda| < 100$ and vorticity numbers up to ten, finding always the same qualitative picture. The profiles of the functions m , ϕ and Ω and the energy density T_t^t for typical rotating solutions with the same values of c and Λ are given in Fig. 1.

We can see that, for given c , Λ and n , the asymptotic value M of the metric function $m(r)$, and the particle number N , decrease with the vorticity number. Also, unexpectedly, both M and N decrease as the node number is increased, for the same values of c , Λ and k . The metric functions $m(r)$ and $e^{-\delta(r)}$ are monotonically increasing, since the right-hand side of eqs. (11) and (12) is always nonnegative. Due to the anisotropy of the stress energy tensor, the configurations are differentially rotating, the rotation function $\Omega(r)$ starting with a nonvanishing value at the origin and monotonically decreasing to zero at infinity. For all solutions we have considered, the metric functions are completely regular and show no sign of an apparent horizon. Also we found no ergoregion, and obviously, no causal anomalies, usually associated with rotation.

For given Λ , n and k , we find nontrivial solutions up to a maximal value of c , where the numerical iteration diverges. The value of c_{max} increases with $|\Lambda|$, while the solution with $c = 0$ corresponds to the global AdS_3 . The numerical integration results for M and N are presented in Fig. 2(a) as a function of c , for three distinct values of Λ (note the absence of local extrema for M and N). The variation of M and N with Λ , for fixed values of the parameter c , is shown in Fig. 2(b). The energy of solutions with the same c decreases with $|\Lambda|$ and a divergent result is obtained in the limit $\Lambda \rightarrow 0$. This is an expected result, since a similar behavior has been found for static solutions [23]. It can easily be proven that the rotating term does not affect the nonexistence result found in ref. [23] for BS in asymptotically flat (2+1)-dimensional spacetime.³ Also, we have every reason to believe that the no hair theorem forbidding the existence of static black hole solutions with a harmonically time-dependent scalar field [23] can be generalized for rotating configurations.

The value of $\rho = -T_t^t$ at the origin is nonzero only for $k = 0, \pm 1$ configurations, and vanishes for other vorticity numbers. The energy density is concentrated in an effective mass-circle (for $k = 0, \pm 1$), or in a ring shape, for other values of k (see Fig. 1). Thus, for $|k| > 1$, this situation

³This result can be viewed as a consequence of the absence of self-interaction terms in the scalar field potential. It can be proven that the nonexistence theorem presented in [23] is not valid if V satisfies the condition $\phi \partial V / \partial \phi - 2V < 0$. The field equations have been solved also for a scalar potential on the form $V = \mu^2 \phi^2 (1 + \lambda_1 \phi^2 + \lambda_2 \phi^4)$. For $\lambda_1^2 < 4\lambda_2$, this scalar potential is known to present flat space nongravitating solutions [15, 17]. However, the qualitative properties of the solutions we found are rather similar to the non-selfinteracting case. In particular we failed again to find solutions in the limit $\Lambda \rightarrow 0$.

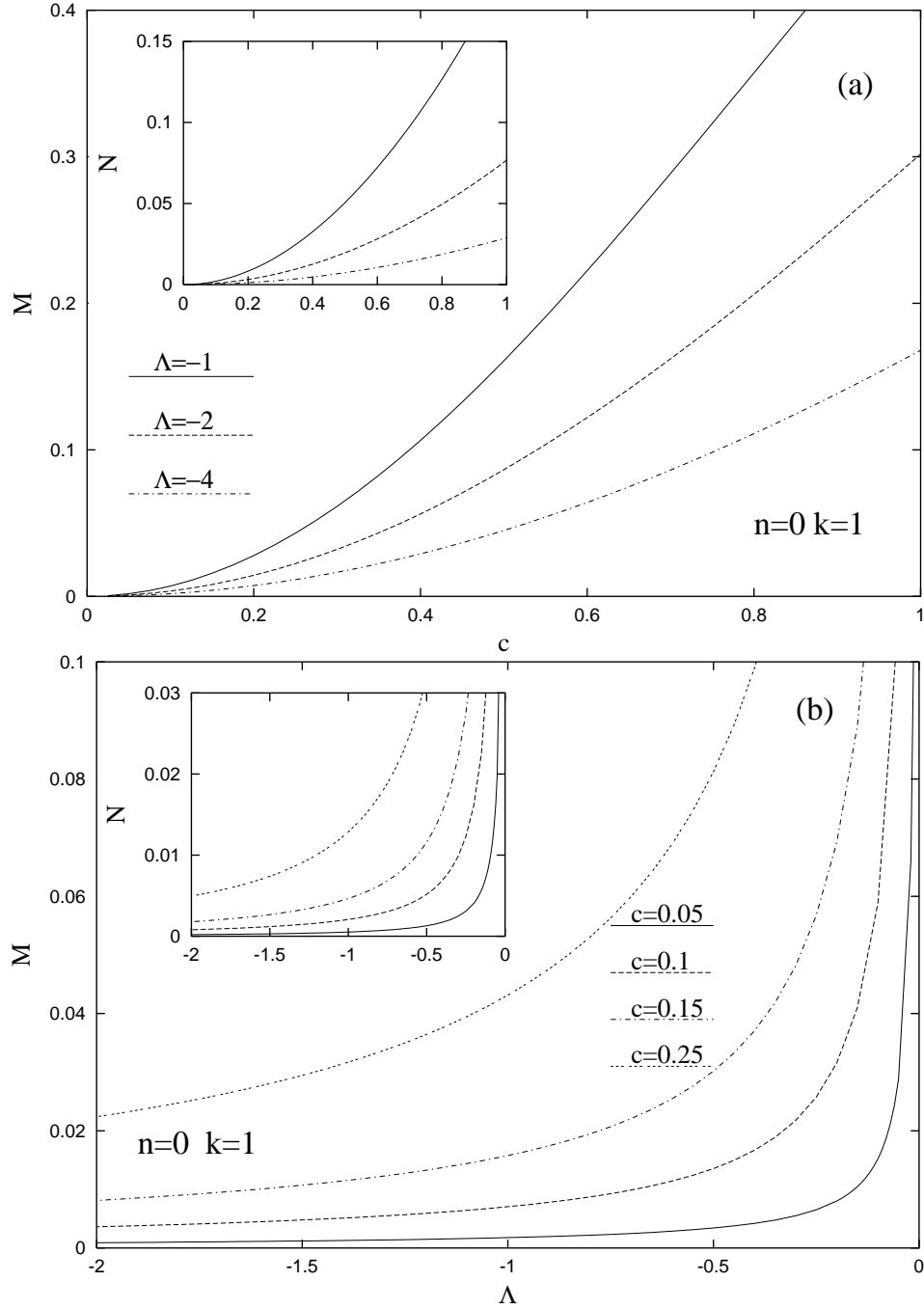


Figure 2. The asymptotic value M of the metric function $m(r)$ and the particle number N are represented as a function of the parameters c and Λ for rotating, nodeless solutions with azimuthal number $k = 1$.

resembles the large self-interaction BS configuration, where a vacuum hole at the center is always present [22]. Some properties of these solutions are better understood by studying the

timelike and null geodesic motion. The radial geodesic equation is

$$\dot{r}^2 = -\varepsilon F + e^{2\delta}(E + \Omega L)^2 - \frac{F}{r^2}L^2 = -V(r), \quad (18)$$

where a dot stands for a derivative with respect to the parameter τ , and $\varepsilon = 1$ or 0 for timelike or null geodesics, respectively. τ is an affine parameter along the geodesics; for timelike geodesics, τ is the proper time. L and E are two constants of motion associated with the Killing vectors ∂_φ and ∂_t , respectively.

The rotational motion ($L \neq 0$) is determined by the centrifugal force term FL^2/r^2 , which forbids a test particle to access the BS center. Therefore any allowed rotational motion has a minimal allowed value of the radial coordinate. The radial motion ($L = 0$) of a massless test particle is unbounded, with a speed $dr/d\tau = E$ at spatial infinity, while $r = 0$ is approached for some finite value of τ . The trajectory of massless particles with $L \neq 0$ is bounded for $L^2/l^2 > E^2$ only. All possible motions of a massive particle are bounded, since $V(r)$ is approximated for large r as

$$V(r) \sim 1 - 2M + \frac{r^2}{l^2} + \frac{L^2}{l^2} - E^2 + O(1/r^2). \quad (19)$$

Massive particles with $L = 0$ can approach the central region of the BS for $E^2 > e^{-2\delta(0)}$ only.

4 The mass and angular momentum

The total mass \bar{M} and angular momentum \bar{J} of (2+1)-dimensional BS solutions with large self-interaction, discussed in refs. [20, 22], have been computed by using a matching procedure on a surface $r = r_0$ separating the regions where the internal (BS region) and external geometries are defined. The external geometry (with a vanishing scalar field) is taken to be the BTZ black hole which gives the values of \bar{M} and \bar{J} . However, in the study of numerical solutions it is desirable to avoid this method and, similar to the (3+1)-dimensional case, to consider a scalar field extending to infinity.

In computing quantities like mass and angular momentum one usually encounters infrared divergences, which are regularized by subtracting a suitably chosen background [24]. However, such a procedure does not work in general; in certain cases the choice of reference background is ambiguous or unknown. In order to regularize such divergences, a different procedure has been proposed in ref. [27]. This technique was inspired by the AdS/CFT correspondence and consists in adding suitable counterterms I_{ct} to the action. These counterterms are constructed from curvature invariants of the induced boundary metric h_{ab} — they contribute an extra surface integral to the action and, obviously, the bulk equations of motion are not altered. As originally found in ref. [27], for vacuum solutions with a negative cosmological constant, the following counterterms are sufficient to cancel divergences in three dimensions

$$I_{ct} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^2x \sqrt{-h} \frac{1}{l}. \quad (20)$$

The boundary divergence-free stress tensor is given by

$$T_{ab} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h^{ab}} = \frac{1}{8\pi G} (K_{ab} - Kh_{ab} - \frac{1}{l} h_{ab}). \quad (21)$$

The efficiency of this approach has been demonstrated in a broad range of examples, the counterterm subtraction method being developed on its own interest and applications.

If there are matter fields on \mathcal{M} , additional counterterms may be needed to regulate the action. In our case, since the scalar field behaves at infinity like $O(r^{-2-\epsilon})$, the counterterm given in eq. (20) is enough to yield a finite result for the boundary stress tensor (see, however, ref. [28] for an example with a different asymptotic behavior, where $\phi \sim O(r^{-1/2})$ and scalar field boundary counterterms I_{ct}^s should be introduced in the action). Using the asymptotic expressions (17), we find the following boundary stress tensor:

$$8\pi GT_{\varphi\varphi} = l(M - \frac{1}{2}) + O(1/r), \quad 8\pi GT_{\varphi t} = \frac{J}{l} + O(1/r), \quad 8\pi GT_{tt} = \frac{1}{l}(M - \frac{1}{2}) + O(1/r). \quad (22)$$

The conserved charges can be constructed by choosing an ADM foliation of ∂M , with spacelike surfaces Σ , so that

$$h_{ab}dx^a dx^b = -N_\Sigma^2 dt^2 + \sigma(d\varphi + N_\Sigma^\varphi dt)^2. \quad (23)$$

In this approach, the conserved quantities associated with a Killing vector of the boundary ξ^b , are given by

$$Q(\xi) = \int_\Sigma d\varphi \sqrt{\sigma} u^a T_{ab} \xi^b, \quad (24)$$

where u^b is the unit timelike vector normal to Σ . The result we find for the mass is $\bar{M} = (2M - 1)/8G$ and is always larger than $-1/8G$, with the extreme value $\bar{M} = -1/8G$ corresponding to the global AdS_3 space. Contrary to what happens for BTZ black holes, we do not find a mass gap between $\bar{M} = -1/8G$ and $\bar{M} = 0$. The total angular momentum \bar{J} of these rotating BS solutions is the charge associated with the Killing vector ∂_φ and has the value $J/4G$.

5 Remarks on charged rotating boson stars

The (2+1)-dimensional case is also an ideal arena to study more complicated rotating configurations, which are very difficult to discuss in higher dimensions. An obvious example is to include a $U(1)$ gauge field in the action principle and to look for charged rotating BS (the static, four-dimensional counterparts of these configurations are found in ref. [29]). Here, the total angular momentum includes a supplementary gauge field contribution which may lead to a violation of the generic relation (9). The electromagnetic field lagrangean $L_{em} = -1/4 F_{ij} F^{ij} \sqrt{-g}$, (with $F_{ij} = \partial_i A_j - \partial_j A_i$), is introduced in the action principle (1), while in the general expresions involving the scalar field, the ordinary derivative is replaced by gauge derivative

$\partial_j \Phi \rightarrow D_j \Phi = \partial_j \Phi + ieA_j \Phi$, e being the gauge coupling constant. The electromagnetic field equations read

$$\nabla_j F^{jl} = eJ^l, \quad (25)$$

while the expression (4) of the energy-momentum tensor includes a supplementary $U(1)$ contribution

$$T_{ij} = D_i \Phi^* D_j \Phi + D_j \Phi^* D_i \Phi - g_{ij} \left(g^{lm} D_l \Phi^* D_m \Phi + V(|\Phi|^2) \right) + F_{il} F_{jm} g^{lm} - \frac{1}{4} g_{ij} F_{lm} F^{lm}. \quad (26)$$

Here we notice the existence of an intriguing expression for the volume integral (10), the contribution of the electromagnetic field to the total angular momentum admitting a representation as a surface integral at infinity. For the same form of the scalar field and an $U(1)$ field ansatz

$$A = A_\varphi d\varphi + A_t dt, \quad (27)$$

the component of the energy-momentum tensor associated with rotation reads

$$T_\varphi^t = D_\varphi \Phi^* D^t \Phi + D_\varphi \Phi D^t \Phi^* + F_{\varphi l} F^{tl} = kJ^t + \frac{1}{\sqrt{-g}} \partial_j \left(A_\varphi F^{jt} \sqrt{-g} \right), \quad (28)$$

k being again the vorticity number and we used also the Maxwell equations (25). This leads to the simple result

$$\int T_\varphi^t \sqrt{-g} = kN + \oint_\infty A_\varphi F^{jt} dS_j, \quad (29)$$

where N is the particle number.⁴ However, different from the pure scalar field case, this integral cannot be identified with the (finite-)total angular momentum. The charged rotating BS solutions asymptotically approach, to leading order, the rotating $U(1)$ BTZ solutions discussed in [30, 31]. The solution of the field equations (25) as $r \rightarrow \infty$ is $A_t \sim Q_e \log(r)$, $A_\varphi \sim C \log(r)$ which modify the asymptotic behavior of the metric functions $m(r)$ and $\Omega(r)$, leading to logarithmic terms (the scalar field still decays as $r^{-2-\epsilon}$). The presence of divergences at spatial infinity in the mass and angular momentum is an usual feature in electrically charged AdS_3 solutions. The mass and angular momentum of charged rotating BS configurations can be computed by using the methods developed for rotating $U(1)$ BTZ case [31, 32]. Here we note that the counterterm formalism does not work in this case, and it is necessary to include a supplementary I_{ct}^{em} term in the general action.

Numerical solutions describing charged rotating BS can easily be obtained by using the same approach presented above. We have integrated the field equations for several values of the cosmological constant, looking for configurations approaching at infinity the charged rotating BTZ background. Static charged solutions with $k = A_\varphi = 0$ have been found as well. However, the presence of an electromagnetic field complicates very much the general picture (for

⁴Note also that the equations (25) implies the relation $Q_e = eN$ between the electric charge of solutions and the particle number.

example, there are two possible sets of boundary conditions at $r = 0$). The solution properties depend also on the value of the coupling constant e and differ significantly from the known four dimensional static charged boson stars. A study of these solutions, as well as a computation of their mass and angular momentum will be presented elsewhere.

The result (29) can easily be generalized to higher dimensions, for values of the cosmological constant $\Lambda \leq 0$ and any scalar field potential (it holds also in the absence of gravity as well). In the more interesting four dimensional case, we expect to find asymptotically a rotating Kerr-Newmann-(AdS) configuration, with $A_\varphi \sim P \cos \theta$, $F^{rt} \sim Q/r^2$ (P and Q corresponding to magnetical, respectiv electrical charge) which yields a finite angular momentum

$$\bar{J} = kN. \quad (30)$$

Thus, the full angular momentum is carried by the complex scalar field, and, similar to the nonabelian dyons [2, 3], the total angular momentum of the gauge field is zero. The construction of such charged rotating four dimensional BS configurations represents a difficult challenge.

6 Conclusions and discussion

We have presented arguments that a gravitating complex scalar field model in (2+1) dimensions admits, in the case of a negative cosmological constant, a continuous family of regular rotating solutions. Here, numerical solutions are already known in the literature in the presence of a self-interaction term $\lambda|\Phi|^4$, as $\lambda \rightarrow \infty$ [22]. However, this limit makes obscure the influence of a number of physical parameters on the solutions properties, and imposes a rather unnatural assumption of scalar field confinement in a finite region of spacetime.

The configurations we have found can be regarded as the lower-dimensional counterparts of the well-known rotating BS solutions in (3+1)-dimensions [7, 8], presenting a number of common properties. However, they are asymptotically AdS, whereas, without self-interaction, no regular solutions are found in the limit $\Lambda \rightarrow 0$. The cosmological constant acts here as an attractive gravitational force, increasing with the radial distance, that is balanced by the scalar field pressure and the centrifugal force. These solutions can be labeled by $(\phi^{(|k|)}(0), n, k)$, where $\phi^{(p)}(0)$ is the p -th derivative of the scalar field evaluated at the origin, n is the node number of the scalar field and k is the vorticity number. The solutions with $k \neq 0$ can be regarded as spinning excitations of the fundamental $k = 0$ static solutions. Similar to other known rotating solitons, their angular momentum is uniquely determined by the value of particle number.

To address the question of stability⁵, we consider the binding energy of these BS solutions, whose natural definition in (2+1) dimensions is [20]

$$E_b \equiv \bar{M} + \frac{1}{8G} - \mu N. \quad (31)$$

⁵A study of stability of a D -dimensional static BS solutions with $\Lambda \leq 0$ under linear perturbations was presented in ref. [23], leading to a Sturm–Liouville-type eigenvalue problem. However, the significance of these results in (2+1) dimensions is unclear.

Similar to higher dimensions, this quantity must be negative for stable configurations. The situation we find here is more complicated, depending on the value of cosmological constant. For small values of $|\Lambda|$, E_b is negative for all allowed values of c . Once $|\Lambda|$ is increased, we notice the $E_b < 0$ for $c > c_0$ only, while $c_0 \rightarrow c_{max}$ for large enough $|\Lambda|$ (for example, the $\Lambda = -1$ configurations presented in Fig.2(a) have $E_b < 0$, those with $\Lambda = -4$ have positive binding energy, while the $\Lambda = -2$ solutions with $c > 0.95$ have $E_b < 0$).

We expect to obtain a very similar qualitative behavior of the solutions when discussing a number of possible extensions of this theory, *e.g.* including a dilaton or a nonminimal coupling term $\xi \Phi^* \Phi R$ term between the scalar field and gravity.

Since a complex scalar field is present in many supergravity theories, one may ask about the possible relevance of BS solutions within the holographic principle and its AdS/CFT correspondence realization. A scalar field has been discussed in this context by many authors, however without considering this type of "macroscopic quantum states" [13].

The (2+1)-dimensional case is rather special, since we could have renormalizable pure Einstein gravity on AdS_3 that can be written as a Chern-Simons theory. The background metric upon which the dual field theory resides is just the rescaled boundary metric

$$\gamma_{ab} = \lim_{r \rightarrow \infty} \frac{l^2}{r^2} h_{ab}, \quad (32)$$

giving the line element $\gamma_{ab} dx^a dx^b = -dt^2 + l^2 d\varphi^2$, which, for a BS solution corresponds to a static cylinder (note also that for a rotating black hole in the bulk, the boundary is rotating [33]). There is now considerable evidence that the boundary conformal field theory corresponding to the bulk Chern-Simons theory is a Liouville field theory (see, *e.g.*, ref. [34]). After canonical quantization, the spectrum of Liouville theory comprises of two different classes [35]: macroscopic (normalizable) states and microscopic (non-normalizable) states. The normalizable states of Liouville theory give a CFT with a central charge $c_{eff} = 1$ (see, *e.g.*, refs. [36, 37]) and so, one expects bulk solutions without horizon would correspond to macroscopic Liouville states.⁶

The presence of additional matter fields in the bulk will not qualitatively change this picture: the Virasoro algebra is an asymptotic isometry of AdS_3 but the boundary theory is not Liouville theory. However, more generally, one can conjecture that any consistent quantum gravity on AdS_3 is dual to a two-dimensional CFT living on the boundary. In light of the AdS/CFT correspondence, Balasubramanian and Kraus have interpreted eq. (21) as giving the expectation value of the dual theory stress tensor $\langle \tau^{ab} \rangle = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{eff}}{\delta \gamma_{ab}}$. The relation between $\langle \tau^{ab} \rangle$ and the boundary stress-tensor is [39]

$$\sqrt{-\gamma} \gamma^{ab} \langle \tau_{bc} \rangle = \lim_{r \rightarrow \infty} \sqrt{-h} h^{ab} T_{bc}. \quad (33)$$

Different bulk configurations can have the same asymptotia (are locally AdS) and so the expectation value of the dual CFT stress tensor is insufficient to distinguish between them (a

⁶A recent proposal to explain BTZ black hole entropy by considering the non-normalizable modes in the boundary Liouville theory can be found in ref. [38].

leading order expression similar to eq. (22) is found also for a BTZ black hole). However, the obstructions of extending the solutions in the bulk will be resolved by additional CFT data.

Here we remark that a scalar field in the bulk has both kind of modes: normalizable and non-normalizable. It is clear now that boundary conditions at infinity and initial conditions for the bulk fields are essentially in this picture. Sources in the dual CFT determine an asymptotic expansion of the corresponding field near the boundary, in other words the non-normalizable bulk modes are equivalent with local operator insertions on the boundary. On the other hand, the normalizable modes are fluctuating in the bulk (for fixed boundary conditions) and quanta occupying such modes have a dual description in the boundary Hilbert space [40, 41]. Boson star is an example of soliton whose field depends on time and does not have a non-trivial topology: the existence and stability of the soliton are governed by its charge, N — in the language of particles, there would have to be N charged particles. In analogy with the pure gravity case, it would be interesting to find an interpretation of boson stars (macroscopic quantum states without horizon) as dual macroscopic (zero temperature) CFT states described by acting on the vacuum with modes of the appropriate boundary operator [41].

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